

Extrapolation, a technique to estimate*

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Abstract

We introduce a technique to estimate a linear operator by embedding it in a family A_t of operators, $t \in (\sigma_0, \infty)$, with suitable curvature properties. One can then estimate the norm of each A_t by bounds that hold in the limit $t \rightarrow \sigma_0$, respectively, $t \rightarrow \infty$. We illustrate this technique on an extension problem that arises in complex geometry.

1 Introduction

This paper grew out of a joint paper with Berndtsson, that used Berndtsson's theorem on the curvature of direct images to prove an Ohsawa–Takegoshi type extension result for holomorphic functions, [BL, B, OT]. Here we distill from [BL] two abstract theorems on estimating Hilbert and Banach space operators by a technique we call extrapolation.

The related notion of interpolation is a well established method in harmonic analysis to estimate Banach space operators. Loosely stated, given two pairs of Banach spaces and operators

$$A_0 : E_0 \rightarrow F_0, \quad A_1 : E_1 \rightarrow F_1$$

among them, one can construct interpolating families E_t, F_t of Banach spaces and operators $A_t : E_t \rightarrow F_t$, $t \in (0, 1)$, whose norms can be bounded in terms of the norms of A_0 and A_1 . There are several inequivalent ways to construct the interpolation spaces. In the complex method, going back to M. Riesz and Thorin, [R, T], the estimate on $\|A_t\|$ is obtained from Hadamard's Three Circles Theorem, i.e., from properties of subharmonic functions.

In extrapolation—again loosely stated—we will be given only one operator, through which we will estimate an entire family of operators $A_t : E_t \rightarrow F_t$. This again depends on subharmonicity and convexity, derived from curvature properties of the bundles that the E_t, F_t form.

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To be concrete, consider two hermitian holomorphic vector bundles (E, h) and (F, k) over the same base S , and a holomorphic homomorphism $A : E \rightarrow F$. We assume the metrics h, k are of class C^2 . The fibers of E and F could be Hilbert spaces, but finite rank bundles will already illustrate the main idea. By the norm $\|A\| : S \rightarrow [0, \infty)$ of A we mean the function $\|A\|(s) = \text{operator norm of } A_s$, where $A_s = A|_{E_s} : (E_s, h) \rightarrow (F_s, k)$. In [L] we have defined what it means for A to decrease curvature; this definition will be reproduced in section 2 and Theorem 3.1.

Theorem 1.1. (i) *If $A : E \rightarrow F$ decreases curvature, then $\log \|A\|$ is plurisubharmonic.*
(ii) *If in addition S is a half plane*

$$S = \{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\},$$

$\|A\|$ is bounded, and $\|A\|(s) = \|A\|(\operatorname{Re} s)$ for $s \in S$, then $\|A\|(t)$ is a decreasing function of $t \in (\sigma_0, \infty)$.

So, in the setting of (ii), if we can estimate $\|A\|(t)$ for some $t = t_0$, the same estimate will hold for $t \geq t_0$, an instance of extrapolation. For this conclusion we had to know in advance that $\sup_S \|A\| < \infty$. Thus (ii) has the flavor of an a priori estimate: assuming a crude bound on all $\|A\|(t)$, a sharper bound follows.

For the application we have in mind we will need an analogous result but with a homomorphism A that increases curvature. We could formulate a clean theorem for quite special bundles E, F . First off, both bundles will be trivial, $E = F = S \times V \rightarrow S$. In this case their metrics h, k , can be thought of as a families h_s, k_s of, say, Hilbertian norms on V . While from the theorem below (F, k) has disappeared, it is really about the norm of the identity homomorphism $A : (F, k) \rightarrow (E, h)$, when k_s is independent of s (and so (F, k) is flat).

Theorem 1.2. *Suppose that $S = \{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\}$ and $E : S \times V \rightarrow S$ is a trivial Hilbert bundle, with $\|\cdot\|$ the norm on V . Suppose further that a C^2 metric h on E has semipositive curvature, $h_s = h_{\operatorname{Re} s}$, and with some $c > 0$*

$$\inf_{s \in S} h_s(v) \geq c\|v\| \quad \text{for all } v \in V.$$

Then $h_t(v)$ is an increasing function of $t \in (\sigma_0, \infty)$ for any $v \in V$, in particular, $h_t(v) \leq \lim_{\tau \rightarrow \infty} h_\tau(v)$.

In section 3 we will formulate and prove results more general than Theorems 1.1 and 1.2, that apply to metrics and norms that are not necessarily Hilbertian, see Theorems 3.1, 3.2.

Various problems in analysis of course boil down to estimating a norm on a space V . The way to do this by extrapolation is to include the norm in a family h_s of norms, $s \in S$, that satisfies the assumptions of say, Theorem 1.2, and estimate h_s in the limit $\operatorname{Re} s \rightarrow \infty$. Whatever estimate one can prove in the limit will then hold for the norm we started with. The success of this approach depends on whether h_s gets easier to estimate as $\operatorname{Re} s \rightarrow \infty$; in the application in section 4 this will be the case.

The method is reminiscent of the continuity method. In the continuity method one attacks a problem P by first connecting it, through a family P_t , $t \in [0, 1]$, $P_0 = P$,

of problems with a problem P_1 that one knows how to solve, and then studying the parameter values t for which P_t can be solved. In this approach the choice of P_t connecting P and P_1 is largely arbitrary, and the success of the method depends on proving suitable estimates for the solutions of P_t . By contrast, it is estimates that extrapolation furnishes. One still needs to connect problem P through a family P_t with a limiting problem $\lim_{t \rightarrow \infty} P_t$ one knows how to solve, but the family P_t should have suitable curvature properties and in its choice is quite restricted.

In the 1980s Rubio de Francia introduced a different notion of extrapolation of operators and estimates, see [R1, R2]. Knowing that a certain operator is bounded in weighted L^p spaces, for fixed $p \in [1, \infty)$ but simultaneously for a large class of weights, he concludes that the operator is bounded in correspondingly weighted L^q spaces for all $q \in [1, \infty)$. The two techniques do not seem to have anything in common.

2 Background

A holomorphic Banach bundle is a holomorphic map $\pi : E \rightarrow S$ of complex Banach manifolds, with each fiber $E_s = \pi^{-1}\{s\}$ endowed with the structure of a complex vector space. It is required that for each $s_0 \in S$ there be a neighborhood $U \subset S$, a complex Banach space W , and a biholomorphic map $\Phi : U \times W \rightarrow \pi^{-1}U$ (a local trivialization) that maps $\{s\} \times W$ to E_s linearly, $s \in U$. If W can be chosen a Hilbert space, we speak of a holomorphic Hilbert bundle.

A metric on E is a locally uniformly continuous function $p : E \rightarrow [0, \infty)$ that restricts on each fiber E_s to a norm p_s inducing the topology of the fiber. We measure the curvature of a metric as follows, see also [L]. First, if $D \subset \mathbb{C}$ is open, $z_0 \in D$, and $u : D \rightarrow \mathbb{R}$ is upper semicontinuous, we let

$$(2.1) \quad \Lambda u(z_0) = \limsup_{r \rightarrow 0} r^{-2} \left(\int_0^1 u(z_0 + re^{2\pi i \theta}) d\theta - u(z_0) \right) \in [-\infty, \infty].$$

In particular, $\Lambda u = \partial^2 u / \partial z \partial \bar{z}$ when $u \in C^2(D)$. Second, if S is a complex manifold, $u : S \rightarrow \mathbb{R}$ is upper semicontinuous, and $\xi \in T^{1,0}S$, we let

$$\xi \bar{\xi} u = \inf \Lambda(u \circ f)(0) \in [-\infty, \infty],$$

where the inf is taken over holomorphic maps f of some neighborhood of $0 \in \mathbb{C}$ into S that send $\partial/\partial z \in T_0^{1,0}\mathbb{C}$ to ξ .

Third, returning to a holomorphic Banach bundle $E \rightarrow S$ endowed with a metric p , we define the Kobayashi curvature $K_\xi(v)$ of p , for $\xi \in T_s^{1,0}S$, $v \in E_s \setminus \{0\}$, by

$$(2.2) \quad K_\xi(v) = -\inf \xi \bar{\xi} \log p(\varphi)(s),$$

the inf taken over sections φ of E , holomorphic near s , such that $\varphi(s) = v$. For example, by [L, Lemma 2.3] $K_\xi(v) \leq 0$ for all ξ, v if and only if $\log p(\varphi)$ is plurisubharmonic for all local holomorphic sections of E . (The above definition (2.2) of curvature differs from [L, (2.6)] by a factor of 2.) If E is a Hilbert bundle, p is the metric associated

with a hermitian metric h of class C^2 , and R is the curvature operator of h (so it is an $\text{End } E$ valued $(1,1)$ form on S), then

$$K_\xi(v) = \frac{h(v, R(\xi, \bar{\xi})v)}{2h(v, v)}.$$

The duals E_s^* of the fibers E_s of a holomorphic Banach bundle also form a holomorphic Banach bundle, denoted E^* . If E was locally isomorphic to $U \times W$, then E^* will be locally isomorphic to $U \times W^*$. If E is endowed with a metric p , the norms p_s^* on E_s^* , dual to the norms p_s , form a metric on E^* , denoted p^* . Let K^* stand for the Kobayashi curvature of p^* .

From now on we assume $\dim S < \infty$.

Lemma 2.1. *If the Kobayashi curvature of (E^*, p^*) is semipositive ($K^* \geq 0$), then the Kobayashi curvature of (E, p) is seminegative ($K \leq 0$).*

When p is a Hilbertian metric, the converse is also true, but we do not know whether this converse holds in general. We also do not know whether in general $K^* \leq 0$ is equivalent to $K \geq 0$.—For the proof of the lemma we need a characterization of plurisubharmonicity:

Lemma 2.2. *Suppose $w : S \rightarrow \mathbb{R}$ is upper semicontinuous, and for every $\xi \in T^{1,0}S$ there is a holomorphic map f of a neighborhood of $0 \in \mathbb{C}$ into S , sending $\partial/\partial z \in T_0^{1,0}\mathbb{C}$ to ξ , such that*

$$(2.3) \quad \Lambda(w \circ f)(0) \geq 0.$$

Then w is plurisubharmonic.

In [L] we already used a similar result. In Lemma 2.3 there the assumption was that (2.3) holds for all holomorphic maps f , which allowed us to quote a theorem of Saks, [Sa]. It turns out that Saks's proof, through a maximum principle, can be tweaked to prove the stronger Lemma 2.2.

Proof. We can assume that S is an open subset of some \mathbb{C}^m . It follows from (2.1) that if $w \in C^2(S)$ then $\Lambda(w \circ f)(0) = \xi \bar{\xi} w = \partial \bar{\partial} w(\xi, \bar{\xi})$, and the limsup there is a limit. Of course, the point of Lemma 2.2 is that w is just assumed upper semicontinuous. However, upon replacing $w(s)$ by $w(s) + \varepsilon|s|^2$, with $\varepsilon > 0$, we can assume without loss of generality that whenever $\xi \in T^{1,0}S$ is nonzero, in (2.3) the strict inequality

$$\Lambda(w \circ f)(0) > 0$$

holds. Consider now a $u \in C^2(S)$ such that for every $s \in S$ there is a nonzero $\xi \in T_s^{1,0}S$ with $\xi \bar{\xi} u = \partial \bar{\partial} u(\xi, \bar{\xi}) = 0$. If f is chosen as in the lemma, then $\Lambda(w \circ f - u \circ f) > 0$, which shows that $w - u$ cannot have a local maximum in S .

To prove that w is plurisubharmonic, we need to take a closed disc $\Delta \subset S$, which we assume to be

$$\Delta = \{s \in \mathbb{C}^m : |s_1| \leq r, \quad s_2 = \cdots = s_m = 0\};$$

a function $h \in C(\Delta)$, harmonic on the open disc, such that $u < h$ on the boundary of Δ ; and prove that this implies $u < h$ on all of Δ . Choose $\delta > 0$ so that

$$P = \{s \in \mathbb{C}^m : |s_1| \leq r, \quad |s_2|, \dots, |s_m| \leq \delta\} \subset S,$$

and define

$$u(s) = h(s_1) + c(|s_2|^2 + \dots + |s_m|^2), \quad s \in P,$$

where $c > 0$ is so large that $w < u$ on ∂P . Then $\xi \bar{\xi} u = 0$ when $\xi = \partial/\partial s_1$. By what we have said above, this implies $w - u$ has no local maximum in $\text{int } P$, and so it is everywhere negative. In particular, $w < u = h$ on Δ , as needed.

Proof of Lemma 2.1. We need to show that $\log p(\varphi)$ is plurisubharmonic for any local holomorphic section φ of E . Again we can assume S is an open subset of some \mathbb{C}^m , and upon replacing p by $pe^{\varepsilon|s|^2}$, $\varepsilon > 0$, that $K_\xi^*(l) > 0$ when $\xi \neq 0, l \neq 0$. Fix $s \in S$ and assume $\varphi(s) = v \neq 0$. By the Banach–Hahn theorem there is a linear form $l \in E_s^*$ of norm 1 such that $\langle l, v \rangle = p(v)$. Let $\xi \in T_s^{1,0} S \setminus \{0\}$. As $K_\xi^*(l) > 0$, there are a section ψ of E^* , holomorphic near s , such that $\psi(s) = l$, and a holomorphic map f of some neighborhood of $0 \in \mathbb{C}$ into S , sending $\partial/\partial z \in T_0^{1,0} \mathbb{C}$ to ξ , these two satisfying

$$(2.4) \quad \Lambda \log p^*(\psi \circ f)(0) \leq 0.$$

At the same time

$$\log |\langle \psi \circ f, \varphi \circ f \rangle| \leq \log p^*(\psi \circ f) + \log p(\varphi \circ f),$$

with equality at $0 \in \mathbb{C}$. Since the left hand side is harmonic near 0, by (2.4)

$$\Lambda \log p(\varphi \circ f)(0) \geq 0.$$

Applying Lemma 2.2 with $w = \log p(\varphi)$ we obtain that $\log p(\varphi)$ is plurisubharmonic wherever it is finite; and this implies it is plurisubharmonic in fact everywhere.

3 The main results

The following results generalize Theorems 1.1 and 1.2:

Theorem 3.1. (i) Consider a holomorphic Hilbert bundle $E \rightarrow S$, endowed with a hermitian metric h of class C^2 , and a holomorphic Banach bundle $F \rightarrow S$ endowed with a metric p . If a holomorphic homomorphism $A : E \rightarrow F$ decreases curvature in the sense that

$$K_\xi^p(Av) \leq K_\xi^h(v) \text{ for all } s \in E_s, \xi \in T_s^{1,0} S, \text{ and } v \in E_s, Av \neq 0,$$

then $\log \|A\|$ is plurisubharmonic.

(ii) In particular, if S is a half plane $S = \{s \in \mathbb{C} : \text{Re } s > \sigma_0\}$, $\|A\|$ is bounded, and $\|A\|(s) = \|A\|(\text{Re } s)$, then $\|A\|(t)$ is a decreasing function of $t \in (\sigma_0, \infty)$.

Proof. Part (i) was proved, in a somewhat greater generality, in [L, Theorem 2.4]; see also the discussion at the end of that paper. When $\text{rk } E < \infty$, this was proved earlier in [CS], again in greater generality. Part (ii) immediately follows: under the assumptions $\log \|A\|(s)$ is subharmonic and independent of $\text{Im } s$, hence $\log \|A\|(t)$ is convex. If it is also bounded above, it must decrease.

Somewhat related convexity theorems, albeit in the framework of interpolation, have already occurred in [H, St].

Theorem 3.2. *Consider a trivial Banach bundle $E = S \times V \rightarrow S$ over a half plane $S = \{s \in \mathbb{C} : \text{Re } s > \sigma_0\}$, endowed with a metric p such that p_s , viewed as norms on V , depend only on $\text{Re } s$. Assume that the second dual p^{**} of p has semipositive Kobayashi curvature $K^{**} \geq 0$. If for all $l \in V^*$, or at least for l in a dense subset of V^* ,*

$$(3.1) \quad \sup \{p_t^*(l) : t \in (\sigma_0, \infty)\} < \infty,$$

then $p_t(v)$ is an increasing function of t , for all $v \in V$. In particular, $p_t(v) \leq \lim_{\tau \rightarrow \infty} p_\tau(v)$.

When p is a Hilbertian metric, and more generally, when V is reflexive, (E^{**}, p^{**}) and (E, p) are isometrically isomorphic, so that the assumption $K^{**} \geq 0$ is the same as $K \geq 0$. But for a general V we could not prove the theorem just assuming $K \geq 0$.

Proof. By Lemma 2.1 the dual metric p^* has seminegative curvature. Hence for any $l \in V^*$ the function $s \mapsto \log p_s^*(l)$ is subharmonic and $\log p_t^*(l)$ is convex in $t \in (\sigma_0, \infty)$. Knowing for a dense set of l 's that the latter function is bounded above implies it decreases, and then by continuity it in fact decreases for all $l \in V^*$.

Let now $v \in V$ and $t > \sigma_0$. By the Banach–Hahn theorem there is an $l \in V^*$ such that $\langle l, v \rangle = p_t^*(l)p_t(v)$. Thus

$$|\langle l, v \rangle| \leq p_\tau^*(l)p_\tau(v) \quad \text{for all } \tau \in (\sigma_0, \infty),$$

with equality when $\tau = t$. Since $p_\tau^*(l) \leq p_t^*(l)$ when $\tau > t$, we must have $p_\tau(v) \geq p_t(v)$. This proves that $p_t(v)$ is an increasing function of t .

4 Application. Extending holomorphic sections

We illustrate the technique of extrapolation, Theorem 3.2, by deriving an Ohsawa–Takegoshi type extension theorem. Ohsawa–Takegoshi type refers to extending from a submanifold Y of a complex manifold X holomorphic sections of certain vector bundles to all of X ; the L^2 norm of the extension should be controled by the L^2 norm of the data, and the constant in this estimate should depend on crude geometric properties of X and Y . Following the original work of Ohsawa and Takegoshi [OT] a great many instances of such theorems were discovered—the list would be too long to reproduce here. In this section we will discuss a version and its proof that is a variant of what is done in [BL]. The reader will be able to trace back to [BL] most of the ideas in our current proof. If there is improvement it is that some of the ideas in [BL] are no longer

needed here. The advantage of this approach is that it is quite painless, and produces arguably sharp estimates. In the setting we treat here we recover the estimates that Blocki and Guan–Zhou obtain in [Bl, GZ] by a different approach.

We will consider the problem to extend sections of the canonical bundle K_X of a Stein manifold X . We could deal in the same way with bundles obtained by twisting K_X by a Nakano semipositive vector bundle, but to keep notation simple we refrain from doing so. For sections f of K_X there is a natural L^2 norm $|\int_X f \wedge \bar{f}|$; however, for sections of the restriction of K_X to a submanifold Y an L^2 norm, which will depend on the geometry of Y , has to be defined.

We start with an oriented smooth manifold X of dimension $a < \infty$, and a submanifold $Y \subset X$ of dimension b . We assume given a function $r : X \rightarrow [0, \infty)$, of class C^3 , whose zero set is Y , and whose critical points on Y are transversely nondegenerate. Let $\mathcal{K} \rightarrow X$ denote bundle of real valued forms of (maximal) degree a . If $\omega \in \mathcal{K}$, we denote by $|\omega|$ either ω or $-\omega$, whichever is a nonnegative multiple of the orientation form. Given any compactly supported continuous section φ of $\mathcal{K}|Y$, we extend it to a compactly supported continuous section of \mathcal{K} , and define an L^1 -type norm of φ by

$$(4.1) \quad \|\varphi\|_1 = \lim_{\varepsilon \rightarrow 0} \varepsilon^{b-a} \int_{\{x \in X : r(x) < \varepsilon^2\}} |\psi|.$$

Near any point of Y we can choose coordinates x_1, \dots, x_a on X so that $r = x_{b+1}^2 + \dots + x_a^2$. If φ and ψ are supported in this coordinate neighborhood U , and $\varphi = f dx_1 \wedge \dots \wedge dx_a$, one easily computes that the limit in (4.1) exists and

$$\|\varphi\|_1 = \sigma \int_{Y \cap U} |f| dx_1 \dots dx_b,$$

independently of the choice of ψ . Here σ is the volume of the unit ball in \mathbb{R}^{a-b} . By a partition of unity it follows that the limit in (4.1) always exists, and is independent of how we choose ψ .

Next, if φ is a general continuous section of $\mathcal{K}|Y$, we define $\|\varphi\|_1 = \sup \|\varphi'\|_1 \leq \infty$, the sup taken over all compactly supported continuous sections φ' such that $|\varphi'| \leq |\varphi|$.

Let $\mathcal{K}_Y \rightarrow Y$ denote the bundle of b -forms on Y . There is a pairing

$$\bigwedge^{a-b} TX|Y \times_Y \mathcal{K}|Y \rightarrow \mathcal{K}_Y,$$

given by contraction and restriction to Y . This pairing descends to a pairing of line bundles

$$\bigwedge^{a-b} N_Y \times_Y \mathcal{K}|Y \rightarrow \mathcal{K}_Y, \quad (\xi, \varphi) \mapsto \iota_\xi \varphi$$

involving the normal bundle $N_Y = (TX|Y)/TY$ of Y . It is easy to see that if Y is also oriented, r determines a continuous section ξ of $\bigwedge^{a-b} N_Y$ such that $\|\varphi\|_1$ can be computed as

$$\|\varphi\|_1 = \int_Y |\iota_\xi \varphi| = \int_Y \iota_\xi |\varphi|.$$

For example, if in local coordinates $r = x_{b+1}^2 + \cdots + x_a^2$, and X, Y are oriented by $dx_1 \wedge \cdots \wedge dx_a$, respectively $dx_1 \wedge \cdots \wedge dx_b$, then ξ will be (the class in $\bigwedge^{a-b} N_Y$ of) $(-1)^{a(a-b)} \sigma \partial / \partial x_{b+1} \wedge \cdots \wedge \partial / \partial x_a$. We will not use this formula, but we will need an alternative representation of the limit in (4.1) in terms of a cut off milder than the sharp cut off $r < \varepsilon^2$. For this purpose we fix a continuous $\chi : \mathbb{R} \rightarrow \mathbb{R}$,

$$(4.2) \quad \chi(t) \begin{cases} = 0, & \text{for } t \leq 0 \\ > 0, & \text{for } t > 0, \end{cases} \quad \liminf_{t \rightarrow \infty} \frac{\chi(t)}{t} > 0.$$

Proposition 4.1. *If ψ is a compactly supported continuous section of \mathcal{K} and φ its restriction to Y (so a section of $\mathcal{K}|Y$), then*

$$(4.3) \quad \|\varphi\|_1 = \lim_{t \rightarrow \infty} e^{(a-b)t/2} \int_X e^{-\chi(t+\log r)} |\psi|.$$

Proof. Writing $X(t) = \{x \in X : r(x) < e^{-t}\}$,

$$e^{(a-b)t/2} \int_{X(t)} e^{-\chi(t+\log r)} |\psi| = e^{(a-b)t/2} \int_{X(t)} |\psi| \rightarrow \|\varphi\|_1$$

as $t \rightarrow \infty$ by (4.1). To estimate the contribution of $X \setminus X(t)$ to the integral in (4.3) we can assume, as before, that ψ is supported in a coordinate chart and in this chart $r = x_{b+1}^2 + \cdots + x_a^2$. With a suitable $M \in \mathbb{R}$ and by a change of coordinates $y = e^{t/2}x$

$$\begin{aligned} e^{\frac{(a-b)t}{2}} \int_{X \setminus X(t)} e^{-\chi(t+\log r)} |\psi| &\leq M e^{\frac{(a-b)t}{2}} \int_{x_{b+1}^2 + \cdots + x_a^2 \geq e^{-t}} e^{-\chi(t+\log(x_{b+1}^2 + \cdots + x_a^2))} dx_{b+1} \cdots dx_a \\ &\leq M \int_{y_{b+1}^2 + \cdots + y_a^2 \geq 1} e^{-\chi(t+t \log(y_{b+1}^2 + \cdots + y_a^2))} dy_{b+1} \cdots dy_a \rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$ by dominated convergence (the point being that when t is large, the last integrand is $\leq (y_{b+1}^2 + \cdots + y_a^2)^{b-a}$ by (4.2)).

Now let $X \supset Y$ be complex manifolds and K_X the canonical bundle of X . A function $r \in C^3(X)$ as above defines an L^2 norm for continuous sections φ of $K_X|Y$,

$$(4.4) \quad \|\varphi\|_Y^2 = \|i^{n^2} \varphi \wedge \bar{\varphi}\|_1 \leq \infty, \quad n = \dim Y.$$

Theorem 4.2. *Let X be a Stein manifold, Y a complex submanifold, $\dim_{\mathbb{C}} X = m$ and $\dim_{\mathbb{C}} Y = n$. Suppose $r : X \rightarrow [0, 1]$ is of class C^3 , vanishes precisely on Y , and its critical points on Y are nondegenerate in directions transverse to Y . If $\log r$ is plurisubharmonic, any holomorphic section f of $K_X|Y$ can be extended to a holomorphic section g of K_X satisfying*

$$(4.5) \quad \left| \int_X g \wedge \bar{g} \right| \leq \|f\|_Y^2.$$

Proof. We will include this extension problem in a family of extension problems P_t to which extrapolation can be applied and which becomes trivial in the limit $t \rightarrow \infty$. The extension problems will be of the same nature as what the theorem is claiming to solve, with the difference that on the left of (4.5) the L^2 norm will be replaced by weighted L^2 norms, with the weights concentrating near Y more and more sharply as $t \rightarrow \infty$. Before constructing this family, though, we exhaust X by a sequence X_ν of relatively compact pseudoconvex subsets. It will then suffice to find $g = g_\nu \in \mathcal{O}(K_{X_\nu})$ extending $f|_{Y \cap X_\nu}$ such that

$$\left| \int_{X_\nu} g \wedge \bar{g} \right| \leq \|f\|_Y^2,$$

because once g_ν found, a subsequence will converge locally uniformly to a $g \in \mathcal{O}(K_X)$ that extends f and satisfies (4.5).

We pick a smooth convex function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (4.2). With ν fixed, for any $s \in \mathbb{C}$ we define a Hilbert norm on a subspace $W \subset \mathcal{O}(K_{X_\nu})$. Let

$$q_s(\psi) = e^{(m-n)\operatorname{Re} s/2} \left| \int_{X_\nu} e^{-\chi(\operatorname{Re} s + \log r)} \psi \wedge \bar{\psi} \right|^{1/2} \leq \infty, \quad \psi \in \mathcal{O}(K_{X_\nu});$$

the Bergman space $W = W_\nu$ consists of $\psi \in \mathcal{O}(K_{X_\nu})$ for which $q_s(\psi) < \infty$ for some (and then for all) $s \in \mathbb{C}$. On W all the norms q_s are equivalent, and

$$q_s(\psi) = e^{(m-n)\operatorname{Re} s/2} \left| \int_{X_\nu} \psi \wedge \bar{\psi} \right|^{1/2}, \quad \text{when } \operatorname{Re} s \leq 0.$$

The norms q_s together define a metric q on the bundle $F = \mathbb{C} \times W \rightarrow \mathbb{C}$. Since the function

$$\mathbb{C} \times X_\nu \ni (s, x) \mapsto (n-m)\operatorname{Re} s + \chi(\operatorname{Re} s + \log r(x))$$

is plurisubharmonic, by Berndtsson's direct image theorem, the Kobayashi curvature of q is semipositive. [B, Theorem 1.1] deals with pseudoconvex subdomains of \mathbb{C}^m instead of Stein X_ν , in which case the canonical bundle is trivial; but the proof carries over to Stein manifolds. Another difference is that [B, Theorem 1.1] claims Nakano semipositivity of the direct image bundle. However, for one dimensional bases like \mathbb{C} Nakano semipositivity is the same as Griffiths and Kobayashi semipositivity (the latter two are equivalent for arbitrary bases as long as the metrics involved are hermitian and of class C^2).

Consider next the closed subspace $I \subset W$ consisting of sections ψ that vanish on $Y \cap X_\nu$, and the quotient $W/I = V = V_\nu$. The trivial bundle $E = \mathbb{C} \times V \rightarrow \mathbb{C}$ is a quotient of F by the subbundle $\mathbb{C} \times I \rightarrow \mathbb{C}$, and inherits a Hilbertian metric p ,

$$p_s(v) = \inf\{q_s(\psi) : \psi \in v\}, \quad v \in V = W/I.$$

Here \inf can be replaced by \min ; the minimum will be attained by $\psi \in v$ that is perpendicular to I , when measured in q_s . As a quotient of a semipositively curved metric, p itself will be semipositively curved, something that is straightforward from the definition of Kobayashi curvature, (2.2). Thus $K^{p^{**}} = K^p \geq 0$. To apply Theorem 3.2 we need to check one more assumption, (3.1). We take an arbitrary $\sigma_0 < 0$ and $S = \{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\}$.

So let $l : V \rightarrow \mathbb{C}$ be a linear form. Composing it with the projection $W \rightarrow V$ we obtain a form $L \in W^*$ whose kernel contains I . Examples of such forms come from sections of $\overline{K}_{Y \cap X_\nu} \otimes T^{m-n,0} X_\nu|Y$ as follows ($\overline{K}_{Y \cap X_\nu}$ stands for the bundle of $(0, n)$ forms on $Y \cap X_\nu$). Given $y \in Y$, $\lambda = \alpha \otimes \xi \in \overline{K}_Y|_y \otimes T_y^{m-n,0} X$, and $\psi \in K_X|_y$, set

$$\lambda \bar{\wedge} \psi = \alpha \wedge \iota_\xi \psi.$$

If now λ is a compactly supported continuous section of $\overline{K}_{Y \cap X_\nu} \otimes T^{m-n,0} X_\nu|Y$, we define $L_\lambda \in W^*$ by

$$L_\lambda(\psi) = \int_{Y \cap X_\nu} \lambda \bar{\wedge} \psi, \quad \psi \in W.$$

Clearly, L_λ vanishes on I , and linear forms of this type are dense among forms in W^* that vanish on I , because $\psi \in W$ must be in I if $L_\lambda(\psi) = 0$ for all λ . In checking (3.1) we can restrict ourselves to $l \in V^*$ induced by such L_λ . Since $p_s^*(l) = q_s^*(L_\lambda)$, it suffices to check

$$\sup_{s \in S} q_s^*(L_\lambda) < \infty$$

for every compactly supported continuous λ . Even better, we may assume that λ is supported in a coordinate patch on X_ν , where Y is given in local coordinates by $x_{n+1} = \dots = x_m = 0$. We take a $\psi \in W$ and apply the submean value theorem to the coefficient in $\psi \wedge \bar{\psi}$ on balls

$$x_1 = \text{const}, \dots, x_n = \text{const}, \quad |x_{n+1}|^2 + \dots + |x_m|^2 < \varepsilon e^{-\text{Re } s},$$

with $\varepsilon > 0$ small but fixed independently of $s \in S$. Letting $\xi = \partial/\partial x_{n+1} \wedge \dots \wedge \partial/\partial x_m$, this gives

$$\left| \int_{Y \cap \text{supp } \lambda} \iota_\xi \psi \wedge \overline{\iota_\xi \psi} \right| \leq C q_s(\psi)^2,$$

C independent of ψ and $s \in S$. Hence $|L_\lambda(\psi)| \leq C' q_s(\psi)$ holds, and so $\sup_{s \in S} p_s^*(l) = \sup_{s \in S} q_s^*(L_\lambda) < \infty$. Therefore Theorem 3.2 applies: for any $v \in V$ and $t > \sigma_0$

$$(4.6) \quad p_t(v) \leq \lim_{\tau \rightarrow \infty} p_\tau(v).$$

This implies the theorem as follows. By Cartan's Theorem B we can extend $f \in \mathcal{O}(K_X|Y)$ to $\psi \in \mathcal{O}(K_X)$. For each ν let $v_\nu \in V_\nu$ be the class of $\psi|X_\nu \in W_\nu$. Choose a compactly supported continuous function $\theta : X \rightarrow [0, 1]$ that is 1 on X_ν . By (4.6), Proposition 4.1, and (4.4)

$$\begin{aligned} p_t(v_\nu)^2 &\leq \lim_{\tau \rightarrow \infty} p_\tau(v_\nu)^2 \leq \limsup_{\tau \rightarrow \infty} q_\tau(\psi|X_\nu)^2 \\ &\leq \limsup_{\tau \rightarrow \infty} e^{(m-n)t} \left| \int_X e^{-\chi(t+\log r)} \theta \psi \wedge \bar{\theta \psi} \right| \leq \|f\|_Y^2. \end{aligned}$$

This means that for every t there is a $g_\nu \in v_\nu$, i.e., $g_\nu \in W_\nu$ extending $f|Y \cap X_\nu$, such that $q_t(g_\nu) \leq \|f\|_Y$. In particular, setting $t = 0$

$$\left| \int_{X_\nu} g_\nu \wedge \bar{g}_\nu \right| \leq \|f\|_Y^2,$$

which, as we have seen, implies the theorem.

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